MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.323: Relativistic Quantum Field Theory I

PROBLEM SET 7
(Complete Version)

REFERENCES: Peskin and Schroeder, Sections 3.1 – 3.4. Informal notes on distributions and the Fourier transform (from the 8.323 website).

Problem 1: A tale of three cutoffs

Consider the function

$$g(t) = \theta(t) e^{-i\omega_0 t} \ . \tag{1.1}$$

In this problem we will examine the Fourier transform of this function, which is defined formally by the ill-defined integral

$$\tilde{g}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t \, e^{i\omega t} \, g(t) \ . \tag{1.2}$$

While this integral does not exist, it should be possible to define the Fourier transform of $g(t)$ in the sense of distributions.

The Fourier transform will then not be a function, but instead will be a distribution, which we will denote generically as $F[\varphi]$, where $\varphi(t)$ is the test function on which the distribution will act. The function $g(t)$ itself is replaced by the distribution

$$F_g[\varphi] \equiv \int_{-\infty}^{\infty} \mathrm{d}t \, g(t) \, \varphi(t) \ . \tag{1.3}$$

(a) Let $F^{(1)}[\varphi]$ denote the Fourier transform of $F_g[\varphi]$, in the formal sense of distributions, which can also be denoted by $\tilde{F}_g[\varphi]$. Evaluate $F^{(1)}[\varphi]$ for the specific choice

$$\varphi(\omega) = \varphi_0(\omega) \equiv \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(\omega-\omega_1)^2/(2\sigma^2)} \ . \tag{1.4}$$

You may leave your answer in the form of a definite integral, or you can evaluate it explicitly in terms of the error function (also called the Fresnel integral)

$$\Phi(x) \equiv \mathrm{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{d}t \, e^{-t^2} \ . \tag{1.5}$$

(b) Another way to give meaning to expression (1.2) is to insert a convergence factor, replacing the original function by

$$g_\epsilon(t) = \theta(t) e^{-i\omega_0 t} e^{-\epsilon t} \ . \tag{1.6}$$

This function can be Fourier-transformed by the usual definition,

$$\tilde{g}_\epsilon(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t \, e^{i\omega t} \, g_\epsilon(t) \ . \tag{1.7}$$

As a function there is no way to compare this with the answer in part (a), but we can promote it to a distribution by defining

$$F^{(2)}[\varphi] \equiv \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \tilde{g}_\epsilon(\omega) \, \varphi(\omega) \ . \tag{1.8}$$

Evaluate this functional for $\varphi_0(\omega)$ given by Eq. (1.4). [*Hint:* you may want to make use of the fact that if an integral is absolutely convergent, you can exchange orders of integration without worry.]

(c) The distribution theory approach guarantees us that we can use any cutoff function that we want, provided only that

$$|g_\epsilon(t)| < g(t) \qquad \text{for } \epsilon > 0 \tag{1.9a}$$

and

$$\lim_{\epsilon \to 0} g_\epsilon(t) = g(t) \qquad \text{for each } t, \tag{1.9b}$$

where the limit is not required to be uniform in t. Another possible cutoff, therefore, would be to simply truncate the integration at $\Lambda = 1/\epsilon$. So let

$$g_\epsilon^{(3)}(t) = \theta(t) \, e^{-i\omega_0 t} \, \theta\left(\frac{1}{\epsilon} - t\right) \ . \tag{1.10}$$

This function can also be Fourier-transformed, in analogy to Eq. (1.7), but the result $\tilde{g}_\epsilon^{(3)}(\omega)$ looks very little like $\tilde{g}_\epsilon(\omega)$. However, we can define the distribution

$$F^{(3)}[\varphi] \equiv \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \tilde{g}_\epsilon^{(3)}(\omega) \, \varphi(\omega) \ . \tag{1.11}$$

Evaluate this functional for $\varphi_0(\omega)$ given by Eq. (1.4). If all goes well you should be able to show that it is equal to the previous two cases.

(d) Small violations of the condition (1.9a) do not usually matter, but if one violates it grossly one can indeed construct cutoff functions $g_\epsilon(t)$ which are still consistent with condition (1.9b), but which lead to distributions that are not equivalent to $F^{(1)}[\varphi]$. Construct such a cutoff function. You need not evaluate $F[\varphi_0]$ for your case, but carry it far enough to show that the answer is different from the previous cases.

Problem 2: A Composite Operator

Since $\phi(x)$ is not an operator, but instead an operator-valued distribution, the quantity $\phi^2(x)$ is not defined. There is no general definition for the square of a distribution. For example, you are probably aware that the square of a delta-function makes no sense. Nonetheless it is possible to define a composite operator $:\phi^2(x):$ which has some, but not all, of the properties that one would naively expect for the square of the operator $\phi(x)$. Note for example that some operator of this sort is necessary to give a quantum treatment to the energy density, which classically is given by

$$T^{00} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{1}{2}m^2\phi^2 \ . \tag{2.1}$$

The other terms lead to similar issues in their definition, but for now we will deal only with the ϕ^2 term.

Starting with

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{a(\vec{p})e^{-ip\cdot x} + a^\dagger(\vec{p})e^{ip\cdot x}\right\} \ , \tag{2.2}$$

it seems natural to define

$$\begin{aligned} :\phi^2(x): \equiv \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\ \Big\{ & a(\vec{p})\,a(\vec{q})\,e^{-i(p+q)\cdot x} \\ & + 2\,a^\dagger(\vec{p})\,a(\vec{q})\,e^{i(p-q)\cdot x} + a^\dagger(\vec{p})\,a^\dagger(\vec{q})\,e^{i(p+q)\cdot x}\Big\} \ . \end{aligned} \tag{2.3}$$

The above expression is obtained by naively squaring the expression in Eq. (2.2), and then normal ordering, which means to move the annihilation operators to the right of the creation operators. (To combine the two cross-terms one must also realize that $\vec{p}$ and $\vec{q}$ are variables of integration, so their names can be interchanged.) Since the commutator of a creation and annihilation operator is a c-number, the normal ordering is equivalent to subtracting a c-number from the expression. The c-number can be viewed as the vacuum expectation value of the expression before normal ordering. The c-number subtraction is infinite, since the commutator is integrated over $\vec{p}$.

The $:\ :$ notation indicates normal ordering, which is essential in defining a ϕ^2 operator that gives finite matrix elements for physical states. To make sense out of Eq. (2.3), however, one must remember that it must not be considered an operator, but rather an operator-valued distribution. Eq. (2.3) is correct, but its interpretation has some subtleties to explore.

(a) Show that

$$\left\langle 0\,\middle|\,:\phi^2(x):\ :\phi^2(y):\,\middle|\,0\right\rangle = 2D^2(x-y) \ , \tag{2.4}$$

where

$$D(x-y) \equiv \langle 0\,|\phi(x)\,\phi(y)|\,0\rangle \ . \tag{2.5}$$

This is a special case of Wick's theorem, which we will learn about in Chapter 4 of Peskin and Schroeder.

(b) Since $:\phi^2(x):$ is an operator-valued distribution, let us integrate it with a smooth weight function to see if we obtain a well-behaved operator. Call the weight function $w(\vec{x})$, and consider the quantity

$$O_3[w] \equiv \int \mathrm{d}^3x\, w(\vec{x})\, :\phi^2(\vec{x},t): \ . \tag{2.6}$$

The normal ordering insures that $\langle 0\,|O_3[w]|\,0\rangle = 0$, so the vacuum variance of $O_3[w]$ is given by

$$\begin{aligned}\sigma^2 = \left\langle 0\left|O_3^2[w]\right|0\right\rangle &= \int \mathrm{d}^3x\, \mathrm{d}^3y\, w(\vec{x})\, w(\vec{y})\, \left\langle 0\left|:\phi^2(\vec{x},t):\ :\phi^2(\vec{y},t):\right|0\right\rangle \\ &= 2\int \mathrm{d}^3x\, \mathrm{d}^3y\, w(\vec{x})\, w(\vec{y})\, D^2(\vec{x}-\vec{y}) \ .\end{aligned} \tag{2.7}$$

For spacelike separations we know from Problem Set 3 that

$$D(x-y) = \frac{m}{4\pi^2 r}\, K_1(mr) \ , \tag{2.8}$$

where $K_1(z)$ denotes a modified Bessel function, as defined for example in Gradshteyn and Ryzhik (*Table of Integrals Series and Products*, Academic Press), and $r^2 = -(x-y)^2$. Use the asymptotic behavior of the modified Bessel function to show that the integral in Eq. (2.7) does NOT converge.

To cure the convergence problem, two steps are necessary. First, to regulate $:\phi^2(x):$ we must smear in time as well as in space. This is the generic case in quantum field theories— the field $\phi(x)$ represents the unusual case in which smearing in space alone is sufficient. So we introduce a smearing function $w(x^\mu)$ for 4-vectors, and define

$$O_4[w] \equiv \int \mathrm{d}^4x\, w(x^\mu)\, :\phi^2(x^\mu): \ . \tag{2.9}$$

The analogue to Eq. (2.7) is then

$$\begin{aligned}\sigma^2 = \left\langle 0\left|O_4^2[w]\right|0\right\rangle &= \int \mathrm{d}^4x\, \mathrm{d}^4y\, w(x^\mu)\, w(y^\mu)\, \left\langle 0\left|:\phi^2(x^\mu):\ :\phi^2(y^\mu):\right|0\right\rangle \\ &= 2\int \mathrm{d}^4x\, \mathrm{d}^4y\, w(x^\mu)\, w(y^\mu)\, D^2(x^\mu-y^\mu) \ .\end{aligned} \tag{2.10}$$

This step alone does not quite solve the problem, because the integral will still have a divergence when x^μ is very near y^μ. The right answer is finite, however, and the ambiguity of the integration shown in (2.10) can be eliminated in several alternative ways.

The ambiguity arises because $\langle 0 \,|\, {:}\phi^2(x^\mu){:}\; {:}\phi^2(y^\mu){:} \,|\, 0\rangle$ was treated in Eq. (2.10) as if it were an ordinary function, while in fact it must be considered a distribution. To understand its definition as a distribution, we can go back to Eq. (2.9). The right-hand side cannot be interpreted as an ordinary integral, because ${:}\phi^2(x^\mu){:}$ is not a function. When we say that ${:}\phi^2(x^\mu){:}$ is a distribution, we mean that it is a recipe for defining an operator for every acceptable test function $w(x^\mu)$. Eq. (2.9) does not define $O_4[w]$ as an integral, but is instead just a symbolic way of saying the $O_4[w]$ is the result of applying the distribution ${:}\phi^2(x^\mu){:}$ to the test function $w(x^\mu)$. The explicit definition of $O_4[w]$ can be obtained by using Eq. (2.3) to replace ${:}\phi^2(x){:}$ on the right-hand side of Eq. (2.9). One can then integrate over x^μ, giving an expression of the form

$$O_4[w] = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2p^0}} \int \frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\sqrt{2q^0}} \left\{ a(\vec{p})\, a(\vec{q})\, \tilde{w}(-q^\mu - p^\mu) + \ldots \right\} , \qquad (2.11)$$

where

$$\tilde{w}(p^\mu) \equiv \int \mathrm{d}^4x\, w(x^\mu)\, e^{ip\cdot x} \qquad (2.12)$$

is the Fourier transform of $w(x^\mu)$, and p^0 and q^0 on the right-hand side are determined by the usual relations $p^0 = \sqrt{\vec{p}^{\,2} + m^2}$, $q^0 = \sqrt{\vec{q}^{\,2} + m^2}$. Note that integrating over x^μ before integrating over $\vec{p}$ and $\vec{q}$ appears to be a change in the order of integration, and such changes are often unjustified when divergent integrals are involved. However, our goal is to *define* the distribution ${:}\phi^2(x^\mu){:}$, which is synonymous with defining $O_4[w]$. Eq. (2.11) is the definition we need, and Eq. (2.3) is correct only in the sense that it is interpreted as shorthand for Eq. (2.11).

(c) Fill in the "..." part of Eq. (2.11).

(d) Use your expression for Eq. (2.11) to express $\langle 0 \,|\, O_4^2[w] \,|\, 0\rangle$ as a convergent integral over two three-vectors $\vec{p}$ and $\vec{q}$, treating $w(x^\mu)$ as an arbitrary test function.

Pedagogical Note: Another way to get the right answer is to use Eq. (2.10), but with an appropriate definition for $D(x^\mu - y^\mu)$ as a distribution. This is the approach normally followed for the Feynman propagator (i.e., the time-ordered product), Eq. (2.59) of Peskin and Schroeder:

$$D_F(x-y) \equiv \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon}\, e^{-ip\cdot(x-y)} \ . \qquad (2.13)$$

As we will see later, amplitudes are calculated by inserting the Feynman propagator into integrals which are always carried out before the limit $\epsilon \to 0$ is taken. This is one way of defining a distribution. The analogous expression for the simple (non-time-ordered) product is a refinement of Peskin and Schroeder's Eq. (2.50):

$$\begin{aligned} D(x-y) = \langle 0 \,|\phi(x)\,\phi(y)|\,0\rangle &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip\cdot(x-y-i\epsilon n)} \\ &= \int \frac{\mathrm{d}^4p}{(2\pi)^3}\delta(p^2-m^2)\theta(p^0)e^{-ip\cdot(x-y-i\epsilon n)} \ , \end{aligned} \tag{2.14}$$

where $n^\mu = (1,0,0,0)$ is a unit vector in the positive time direction. Note that the insertion of the $i\epsilon$ term causes the integral to be absolutely convergent, and it also regularizes the infinity that would otherwise occur for $x = y$. With the $i\epsilon$ in place the integral in Eq. (2.10) becomes absolutely convergent, which allows the integrations to be rearranged to match the answer that you should have found in part (d).

(e) Consider the special case where $w(x^\mu)$ is a normalized Gaussian in both space and time:

$$w(x^\mu) = \frac{1}{\pi^{3/2}a^3}e^{-|\vec{x}\,|^2/a^2}\cdot\frac{1}{\sqrt{\pi}\,b}e^{-(x^0)^2/b^2} \ . \tag{2.15}$$

For fixed b, show that

$$\left\langle 0 \left| O_4^2[w] \right| 0 \right\rangle \xrightarrow[a\to\infty]{} \text{const}\, a^\beta \ , \tag{2.16}$$

where β is a constant that you are to determine. (You need not find the constant of proportionality.) Does β agree with what you found in Problem Set 3, Problem 2, for the variance of a smeared scalar field $\phi(x^\mu)$? Does this agree with what we would expect from the assumption that in the limit of large a, $O_4^2[w]$ samples many independent values of $:\phi^2:$?

(f) One property that one might naively expect for the square of an operator is positivity, but $:\phi^2(x^\mu):$ is not positive. (Heuristically we can imagine that we constructed $:\phi^2:$ by first squaring ϕ, but then we made an infinite subtraction which spoils the positivity.) Show that $:\phi^2:$ is not positive by constructing a state $|\psi\rangle$ such that

$$\langle \psi \,|O_4[w]|\, \psi\rangle < 0 \ , \tag{2.17}$$

using the Gaussian weight function of Eq. (2.15). I would recommend looking for a state $|\psi\rangle$ of the form

$$|\psi\rangle = |0\rangle + \delta\,|\psi_1\rangle \ , \tag{2.18}$$

where δ is an arbitrarily small positive constant, so that only contributions to first order in δ need be considered. You need not evaluate $\langle \psi \,|O_4[w]|\, \psi\rangle$ completely, as long as you show that it is negative.

Problem 3: Back to the Dirac Equation — the Gordon Identity

Problem 3.2 of Peskin and Schroeder.