Note on the Lost Sales Total Cost

There has been some confusion and questions concerning the derivation of the total cost equation when lost sales are considered. Page 12 of the Safety Stock Notes gives an approximate (why is this approximate? Hmmmm) Total Cost of Buffer Stock expression when shortages are lost sales as:

$$TC[R] = C_h C_p(ks + N[k]s) + \frac{C_l D}{Q^*} N[k]s$$

Note that this is only looking at the cost of the safety stock component of the inventory. To consider total costs, one would simply add in the expressions for the ordering, holding, and purchase costs. But for this analysis, only the safety stock component is relevant.

One question is why is the N[k]s term included in the holding cost expression? Recall that N[k]s is the expression for the expected units short. Why should we be charged a holding cost for the units that we fall short? To understand why, recall that this is essentially a (Q,R) system, we will order Q whenever we fall to R level of inventory. So, during a stockout, under a backorder situation, each unit short that is demanded will be replenished by the order when it arrives. This means that at the moment of replenishment, the inventory level will be Q-N[k]s, or, the order level minus the expected units that were not fulfilled during the stock out period. Now, suppose that we are under a lost sales condition. During the stock out period, all unfulfilled demand (which is equal to N[k]s) will not be backordered. So, when the replenishment order arrives the inventory level will be Q. In other words, we will have N[k]s more units of inventory on hand compared to the backorder case. So, we will carry that extra inventory going forward. If this is still confusing, draw the inventory on hand diagrams for both conditions, it might make more sense to you.

Now, we want to find what the optimal service level would be. To do this, the process is the same as before, look at the first derivative with respect to k and set the expression equal to zero. Why k? This will determine our stock out probability and service levels which in turn will give us the optimal R value.

The only term that is tricky is N[k] or the unit loss normal function that tells me the expected number of units short in terms of standard deviations. Recall that

$$N[k] = \int_k^\infty (x_0 - k) f_x(x_0) dx_0$$

The Normal distribution has some nice properties, namely,

$$\int_k^\infty (x_0) f_x(x_0) dx_0 = f_x(k)$$

This allows us to restate N[k] as:

$$N[k] = f_x(k) - kP[x \geq k]$$

where P[x≥k] is the probability that the random variable x is larger than k. We know this value as the probability of a stock out or P[SO]. So let's substitute,

$$TC[R] = C_h C_p (ks + s(f_x(k) - kP[SO])) + \frac{C_l D}{Q^*} s(f_x(k) - kP[SO])$$

Taking derivate with respect to k (noting that d(N[k])/dk = -P[SO]:

$$\frac{d(TC[R])}{dk} = C_h C_p (s - sP[SO]) - \frac{C_l D}{Q^*} sP[SO] = 0$$

Therefore, the optimal stockout probability is:

$$P[SO]^* = \frac{Q^* C_h C_p}{DC_l + Q^* C_h C_p}$$

Once we know the P[SO*] we can find SL and then k* and finally R*. You can use the exact same logic to determine the optimal R for the backorder case as well. Or, if you are so inclined, any percentage mix between the two.

What happens if I lose the customer, not just the sales, going forward? Hmmmm – I asked you to think about this in PS3 question 2 – give it some thought but I am not expecting a full derivation – there are many assumptions that would need to be made.

Exact Approach to Determining Cost-Optimal Reorder Points

On page 13 of the Safety Stock Notes, we show an iterative approach to finding the optimal (Q,R). Up to this point, we have set the Q and then solved for an optimal R. Of course, it is not optimal to do this, just practical in many cases. To find the optimal values of Q and R, we need to solve the total cost equation for both values:

$$TC[Q, R] = \frac{C_O D}{Q} + \frac{C_H C_P Q}{2} + C_H C_P(ks) + \frac{C_B D\, N[k]s}{Q}$$

Taking partial derivatives with respect to Q and R and setting each equal to zero yields two equations which must be solved simultaneously:

$$P[SO]^* = \frac{Q^* C_H C_P}{DC_B} \qquad \text{(Eqn 1)}$$

$$Q^* = \sqrt{\frac{2D(C_O + C_B N[k^*]s)}{C_H C_P}} \qquad \text{(Eqn 2)}$$

The iterative approach, as shown in an example in the Safety Stock Notes performs the following steps.

1. Set Q = Q_W using EOQ.
2. Using the Q, find the P[SO*] using Eqn 1
3. Calculate SL = 1 – P[SO]
4. Find k using the SL from either the Normal tables or from equation
5. Find N[k] using the k from either the Unit Normal Loss tables or equation
6. Find the tentative R = d'+sk
7. You can calculate the TC[cycle stock] and TC[safety stock] at this point and for the first iteration of the example, these are $6,250 and $2,058.50, respectively
8. Using the new N[k] value found in step 5, find a new Q* using Eqn 2.
9. If Q* is equal to (or within tolerance of) Q from step 2, STOP, Otherwise, set Q = Q* and go to Step 2.

In the table on page 13, the final row is simply the percentage improvement from initial case for total costs. This is an easy algorithm to write into Excel.