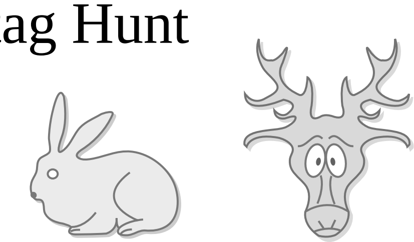
Learning 5
Adjustment with persistent noise
(Kandori, Mailath, Rob)

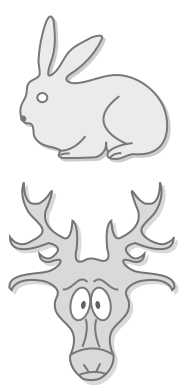
14.126 Game Theory

Haluk Ergin

Muhamet Yildiz

Stag Hunt

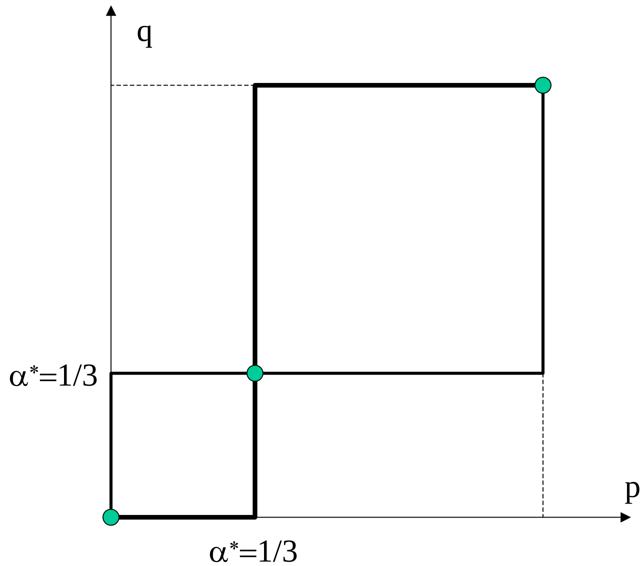




	(2,2)	(4,0)
	(0,4)	(6,6)

Figure by MIT OCW.

Best responses in Stag-Hunt game



Adjustment Process

- N = population size.
- State space: θ_t = # of players who play A at t.
- $u_A(\theta_t) = \theta_t/N\ u(A,A) + (N - \theta_t)/N\ u(A,B)$
- $\theta_{t+1} = P(\theta_t)$, where

$$P(\theta_t) > \theta_t \Leftrightarrow u_A(\theta_t) > u_B(\theta_t)\ \&$$

$$P(\theta_t) = \theta_t \Leftrightarrow u_A(\theta_t) = u_B(\theta_t).$$

- Example:

$$P(\theta_t) = BR(\theta_t) = \begin{cases} N \text{ if } u_A(\theta_t) > u_B(\theta_t) \\ \theta_t \text{ if } u_A(\theta_t) = u_B(\theta_t) \\ 0 \text{ if } u_A(\theta_t) < u_B(\theta_t) \end{cases}$$

Noise

- Independently, each agent with probability 2ε mutates, and plays either of the strategies with equal probabilities.

$$P^{\varepsilon} = \begin{bmatrix} (1-\varepsilon)^N & (1-\varepsilon)^N & \ldots & (1-\varepsilon)^N & \varepsilon^N & \ldots & \varepsilon^N \\ N(1-\varepsilon)^{N-1}\varepsilon & N(1-\varepsilon)^{N-1}\varepsilon & \ldots & N(1-\varepsilon)^{N-1}\varepsilon & N(1-\varepsilon)\varepsilon^{N-1} & \ldots & N(1-\varepsilon)\varepsilon^{N-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ N(1-\varepsilon)\varepsilon^{N-1} & N(1-\varepsilon)\varepsilon^{N-1} & \ldots & N(1-\varepsilon)\varepsilon^{N-1} & N(1-\varepsilon)^{N-1}\varepsilon & \ldots & N(1-\varepsilon)^{N-1}\varepsilon \\ \varepsilon^N & \varepsilon^N & \ldots & \varepsilon^N & (1-\varepsilon)^N & \ldots & (1-\varepsilon)^N \end{bmatrix}$$

- $\varphi^*(\varepsilon)$ = invariant distribution for P^{ε}.

P^{ε}

$$\left[\begin{array}{ccc|ccc} (1-\varepsilon)^N & \cdots & (1-\varepsilon)^N & \varepsilon^N & \cdots & \varepsilon^N \\ N(1-\varepsilon)^{N-1}\varepsilon & \cdots & N(1-\varepsilon)^{N-1}\varepsilon & N(1-\varepsilon)\varepsilon^{N-1} & \cdots & N(1-\varepsilon)\varepsilon^{N-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \binom{N}{N^*-1}(1-\varepsilon)^{N-N^*+1}\varepsilon^{N^*-1} & \cdots & \binom{N}{N^*-1}(1-\varepsilon)^{N-N^*+1}\varepsilon^{N^*-1} & \binom{N}{N^*-1}(1-\varepsilon)^{N^*-1}\varepsilon^{N-N^*+1} & \cdots & \binom{N}{N^*-1}(1-\varepsilon)^{N^*-1}\varepsilon^{N-N^*+1} \\ \hline \binom{N}{N^*}(1-\varepsilon)^{N-N^*}\varepsilon^{N^*} & \cdots & \binom{N}{N^*}(1-\varepsilon)^{N-N^*}\varepsilon^{N^*} & \binom{N}{N^*}(1-\varepsilon)^{N^*}\varepsilon^{N-N^*} & \cdots & \binom{N}{N^*}(1-\varepsilon)^{N^*}\varepsilon^{N-N^*} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon^N & \cdots & \varepsilon^N & (1-\varepsilon)^N & \cdots & (1-\varepsilon)^N \end{array}\right]$$

Invariant Distribution

- $N^* = \lceil \alpha^* N \rceil < N/2$.
- $D_A = \{\theta | \theta \geq N^*\}$; $D_B = \{\theta | \theta < N^*\}$;
- $q_{AB} = \Pr(\theta_{t+1} \in D_A | \theta_t \in D_B)$;
- $q_{BA} = \Pr(\theta_{t+1} \in D_B | \theta_t \in D_A)$;
- $$p_A(\varepsilon) = \sum_{\theta \in D_A} \varphi_\varepsilon^*(\theta)$$
- $p_B(\varepsilon) = 1 - p_A(\varepsilon)$.

Invariant distribution, continued

- $$\begin{bmatrix} p_A(\varepsilon) \\ p_B(\varepsilon) \end{bmatrix} = \begin{bmatrix} 1 - q_{BA} & q_{AB} \\ q_{BA} & 1 - q_{AB} \end{bmatrix} \begin{bmatrix} p_A(\varepsilon) \\ p_B(\varepsilon) \end{bmatrix}$$
- $$\begin{bmatrix} -q_{BA} & q_{AB} \\ q_{BA} & -q_{AB} \end{bmatrix} \begin{bmatrix} p_A(\varepsilon) \\ p_B(\varepsilon) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
- $$\frac{p_A(\varepsilon)}{p_B(\varepsilon)} = \frac{q_{AB}}{q_{BA}}$$

Invariant distribution, continued

$$\frac{p_A(\varepsilon)}{p_B(\varepsilon)} = \frac{q_{AB}}{q_{BA}}$$

$$= \frac{\binom{N}{N^*}\varepsilon^{N^*}(1-\varepsilon)^{N-N^*} + \binom{N}{N^*+1}\varepsilon^{N^*+1}(1-\varepsilon)^{N-N^*-1} + \cdots}{\binom{N}{N^*}\varepsilon^{N-N^*}(1-\varepsilon)^{N^*} + \binom{N}{N^*-1}\varepsilon^{N-N^*+1}(1-\varepsilon)^{N^*-1} + \cdots}$$

$$= \frac{\binom{N}{N^*}\varepsilon^{N^*}(1-\varepsilon)^{N-N^*} + o\left(\varepsilon^{N^*}\right)}{\binom{N}{N^*}\varepsilon^{N-N^*}(1-\varepsilon)^{N^*} + o\left(\varepsilon^{N-N^*}\right)} \cong \frac{1}{\varepsilon^{N-2N^*}} \to \infty.$$

Proposition

If N is large enough so that N* < N/2, then limit φ* of invariant distributions puts a point mass on θ = N, corresponding to all players playing A.