

Problem Set 4

(due on the day of Lecture # 5)

Problem 1 *Augmented Problem (3.10) from Romer*

Suppose that output at firm i is given by $Y_i = K_i^\alpha L_i^{1-\alpha} [K^\phi L^{-\phi}]$, where K_i and L_i are the amounts of capital and labor used by the firm, K and L are the aggregate amounts of capital and labor, and $\alpha > 0, \phi > 0$, and $0 < \alpha + \phi < 1$. Assume that factors are paid their private marginal products; thus $r = \partial Y_i / \partial K_i$. Assume that the dynamics of K and L are given by $\dot{K} = sY$ and $\dot{L} = nL$, and that K_i/L_i is the same for all firms.

1. What is r as a function of K/L ?
 - (a) What is K/L on the balanced growth path?
 - (b) What is r on the balanced growth path?
2. Now suppose there are no externalities in production, that is $Y_i = K_i^\alpha L_i^{1-\alpha}$.
 - (a) Redo 1.
 - (b) How does the balanced growth path K/L (under no externalities) compare to before?
 - (c) Does your analysis support the claim that “if an increase in domestic saving raises domestic investment, positive externalities from capital would mitigate the decline in the private marginal product of capital.” compared to an economy with no externalities from capital? Explain.
3. Compare the marginal product of capital for firm i with the marginal product of capital for a social planner.
4. Relax the assumption that $\alpha + \phi < 1$. Suppose $\alpha + \phi = 1$.
 - (a) What is r in this case?
 - (b) Does an increase in s affect \dot{K}/K along the balanced growth path? Compare to case when $0 < \alpha + \phi < 1$.

Problem 2 *Decentralized Ramsey Model*

Consider an economy composed of many identical families and firms. Families are infinitely-lived with utility functions of the form $U_t = \int_t^\infty u(c_s) e^{-\rho(s-t)} ds$.

Families can save by investing in capital or by lending money to others at the same interest rate. Denote nonhuman wealth per capita by $a_t \equiv k_t - b_t$, where b_t is debt per capita. The dynamic budget constraint is $\dot{a}_t = w_t + r_t a_t - c_t - n a_t$. (wealth per capita accumulates according to wages earned plus interest earned on prior wealth minus what a family consumes, and since it is per capita and there is population growth, the standard na has to be subtracted.) Note that w_t and r_t denote the market wage and interest rate. Each firm acts competitively and produces with a constant returns to scale technology. The aggregate production function is $Y_t = F(K_t, L_t)$

1. Assume that the following No Ponzi Game (NPG) condition holds for the family's optimization problem: $\lim_{t \rightarrow \infty} a_t e^{(-\int_0^t (r_v - n) dv)} = 0$. This is a constraint on how fast debt can grow. It requires that family debt does not increase asymptotically faster than the interest rate. Discount both sides of the dynamic budget constraint by $e^{(-\int_0^t (r_v - n) dv)}$ and integrate the family's budget constraint from time 0 to T. Let T go to infinity and use the NPG condition to express the present value of lifetime consumption as the sum of initial nonhuman wealth a_0 and discounted labor income. This exercise transforms the dynamic budget constraint (DBC) into the intertemporal budget constraint. (Hint: Write the DBC as $\dot{a}_t - (r - n)a_t = w_t - c_t$).
2. Using the dynamic budget constraint, solve the family's maximization problem by setting up a Hamiltonian to obtain an expression for the optimal growth rate of consumption.
3. Derive the first-order conditions for the firm's optimal demand of the two factors of production.
4. Link your results, using the fact that aggregate private debt must equal zero in equilibrium (Why?). Verify that the optimality conditions and the dynamic budget constraint imply that the following two conditions hold:

(a) $f(k_t) = c_t + \dot{k}_t + nk_t$,

(b) $\frac{du'(c_t)/dt}{u'(c_t)} = \rho + n - f'(k_t)$.

How do these conditions compare to those describing the optimal solution to the Central Planner's problem?

5. Introduce a very simple government into the model. There's a fixed level of per-capita government expenditure g at each moment (take this to be exogenous and does not directly affect the marginal utility of consumption of the representative family). The government fully balances its budget at each time with lump sum taxes levied on families. That is, $g_t = \tau_t$, where τ_t is the per capita lump-sum tax.
 - (a) Derive the family's first order conditions.
 - (b) Draw the phase diagram. How do steady state (k^*, c^*) compare to case with no government?