

**Problem Set 3**  
**(due on the day of Lecture # 4)**

**Problem 1** (augmented problem 3.14 from Romer)

Consider the following model with physical and human capital:

$$\begin{aligned} Y(t) &= [(1 - a_K)K(t)]^\alpha [(1 - a_H)H(t)]^{1-\alpha}, \quad 0 < \alpha < 1, \quad 0 < a_K < 1, \quad 0 < a_H < 1, \\ \dot{K}(t) &= sY(t) - \delta_K K(t), \\ \dot{H}(t) &= B[a_K K(t)]^\gamma [a_H H(t)]^\phi [A(t)L(t)]^{1-\gamma-\phi} - \delta_H H(t), \quad \gamma > 0, \quad \phi > 0, \quad \gamma + \phi < 1, \\ \dot{L}(t) &= nL(t), \\ \dot{A}(t) &= gA(t), \end{aligned}$$

where  $a_K$  and  $a_H$  are the fractions of the stocks of physical and human capital used in the education sector.

This model assumes that human capital is produced in its own sector with its own production function. Bodies ( $L$ ) are useful only as something to be educated, not as an input into the production of final goods. Similarly, knowledge ( $A$ ) is useful only as something that can be conveyed to students, not as a direct input to goods production.

1. Define  $k = \frac{K}{AL}$  and  $h = \frac{H}{AL}$ . Derive equations for  $\dot{k}$  and  $\dot{h}$ .
2. Find an equation describing the set of combinations of  $h$  and  $k$  such that  $\dot{h} = 0$ .
3. Differentiate the  $\dot{h} = 0$  equation with respect to  $k$  and  $h$  to find the slope  $\frac{dk}{dh}$ .
4. Calculate  $\frac{d^2k}{dh^2}$  to find the curvature of the  $\dot{h} = 0$  locus. Sketch the locus in  $(h, k)$  space.
5. What are signs of  $\dot{h}$  above and below the  $\dot{h} = 0$  locus? Add the corresponding arrows to your diagram.
6. Repeat steps 2.-5. for the  $\dot{k} = 0$  equation to complete the phase diagram.
7. Does this economy have a balanced growth path? If so, is it unique? Is it stable? What are the growth rates of output per person, physical capital per person, and human capital per person on the balanced growth path?

8. Suppose the economy is initially on a balanced growth path, and that there is a permanent increase in  $s$ . Sketch how the economy moves to the new balanced growth path in your phase diagram. How does this change affect the path of output per person over time? Repeat this analysis for a permanent increase in  $n$ .

**Problem 2** (problem 3.17 from the first edition of Romer's textbook)

Suppose the production function is  $Y(t) = K(t)^\alpha H(t)^{1-\alpha}$  where  $0 < \alpha < 1$ . Notice that it exhibits constant returns to physical and human capital together. Physical capital  $K$  and human capital  $H$  evolve according to  $\dot{K}(t) = s_K Y(t)$  and  $\dot{H}(t) = s_H Y(t)$ .

1. Show that regardless of the initial levels of  $K$  and  $H$  (as long as both are positive), the ratio  $\frac{K}{H}$  converges to some balanced-growth-path level  $(\frac{K}{H})^*$ .
2. Once  $\frac{K}{H}$  has converged to  $(\frac{K}{H})^*$ , what are the growth rates of  $K$ ,  $H$  and  $Y$ ?
3. How, if at all, does the growth rate of  $Y$  on the balanced growth path depend on  $s_K$  and  $s_H$ ?
4. Suppose  $\frac{K}{H}$  starts off at a level that is smaller than  $(\frac{K}{H})^*$ . Is the initial growth rate of  $Y$  greater than, less than, or equal to its growth rate on the balanced growth path?

**Problem 3**

Consider the model of Mankiw, Romer and Weil (1992) presented in class. What values of  $s_K$  and  $s_H$  maximize consumption per unit of effective labor on the balanced growth path?