Transparensies for 14.04

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1 Technology

1.1 Specification

$\mathbf{y} \in \mathbb{R}^n$ – production plan; $y_j = y_j^{\text{output}} - y_j^{\text{input}}$,

$y_j > (<)\ 0$ – net output (input).

Y – production possibilities set, $\mathbf{y} \in Y \subset \mathbb{R}^n$.

1.1.1 Examples:

$\mathbf{y} = (hw, -l)$; $\qquad \mathbf{y} = (hw, -l, -c, -p)$.

Description of a single good (variants):

General: Land,

Flows: Hours per week,

Details: Time, Place, Type.

1.1.2 Single output

$(y, -\mathbf{x}) \in \mathbb{R}_+^{n+1}$, e.g. $(y, -l, -k)$.

Input requirement set:

$$V(y) = \{\mathbf{x} \in \mathbb{R}_+^n : (y, -\mathbf{x}) \in Y\}.$$

Isoquant:

$$Q(y) = \{\mathbf{x} \in \mathbb{R}_+^n : \mathbf{x} \in V(y); \quad \forall y' > y,\ \mathbf{x} \notin V(y')\}.$$

Production function: $f(\mathbf{x}) = \max\limits_{(y,-\mathbf{x})\in Y} y$.

Short-run production function:

$$k = \bar{k},\ y = f(l, \bar{k}) = \bar{f}(l).$$

1.2 Examples

1.2.1 Cobb-Douglas technology

$$Y = \{(y, -x_1, -x_2) \in \mathbb{R}^3 : y \leq x_1^a x_2^{1-a}\},$$

$$Q(y) = \{(x_1, x_2) \in \mathbb{R}_+^2 : y = x_1^a x_2^{1-a}\},$$

$$f(x_1, x_2) = x_1^a x_2^{1-a}.$$

1.2.2 Leontief technology

$$Y = \{(y, -x_1, -x_2) \in \mathbb{R}^3 : y \leq \min(ax_1, bx_2)\},$$

$$Q(y) = \{(x_1, x_2) \in \mathbb{R}_+^2 : y = \min(ax_1, bx_2)\},$$

$$f(x_1, x_2) = \min(ax_1, bx_2).$$

1.3 Assumptions

MONOTONICITY

$$\mathbf{x} \in V(y), \mathbf{x}' \geq \mathbf{x} \Longrightarrow \mathbf{x}' \in V(y).$$

CONVEXITY

$$\mathbf{x}, \mathbf{x}' \in V(y) \Longrightarrow t\mathbf{x} + (1-t)\mathbf{x}' \in V(y), \forall t \in [0,1]$$

or $V(y)$ is a *convex set* ($f(x)$ is *quasiconcave*).

REGULARITY

$V(y)$ is *closed*, *nonempty*.

Note: 1) These are not axioms.

2) Convex Y (problematic assumption) $\Longrightarrow$

convex $V(y)$.

1.4 Technical Rate of Substitution

TRS $\equiv$ Slope of an isoquant.

$$f(x_1, x_2) = \bar{y}, \qquad \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 = 0,$$

$$TRS = \frac{dx_2}{dx_1} = -\frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}}.$$

Cobb-Douglas: TRS $= -\frac{a}{1-a}\frac{x_2}{x_1}$. Leontief: ?

1.5 Elasticity of Substitution

σ measures curvature of an isoquant: how the ratio of factor inputs changes as the slope of the isoquant (TRS) changes. (has to be scale-independent!),

$$\sigma = \frac{d \ln \frac{x_2}{x_1}}{d \ln |\text{TRS}|} = \frac{d\frac{x_2}{x_1}}{\frac{x_2}{x_1}} \frac{|\text{TRS}|}{d|\text{TRS}|}.$$

Cobb-Douglas: $\frac{x_2}{x_1} = -\frac{1-a}{a}\text{TRS}$,

$$\ln \frac{x_2}{x_1} = \ln |\text{TRS}| + \ln \frac{1-a}{a} \Longrightarrow \sigma = 1.$$

Constant Elasticity of Substitution (CES):

for $-\infty \leq \rho \leq 1$

$$y = \left[a_1 x_1^\rho + a_2 x_2^\rho\right]^{\frac{1}{\rho}},$$

$$\text{TRS} = -\left(\frac{x_1}{x_2}\right)^{\rho-1}, \qquad \sigma = \frac{1}{1-\rho} \qquad \text{for } a_1 = a_2.$$

$\rho = 1 \Longrightarrow$ Linear technology,

$\rho = 0 \Longrightarrow$ Cobb-Douglas,

$\rho = -\infty \Longrightarrow$ Leontief.

1.6 Returns to scale

Each input is increased by a factor of t. Output ?.

CONSTANT RETURNS TO SCALE (CRtS):

Technology exhibits CRtS if for all $t \geq 0$,

1. $\mathbf{y} \in Y \Longrightarrow t\mathbf{y} \in Y$; or
2. $\mathbf{x} \in V(y) \Longrightarrow t\mathbf{x} \in V(ty)$; or
3. $f(t\mathbf{x}) = tf(\mathbf{x})$.

INCREASING RETURNS TO SCALE (IRtS):

$$f(t\mathbf{x}) > tf(\mathbf{x}) \text{ for all } t > 1.$$

DECREASING RETURNS TO SCALE:

$$f(t\mathbf{x}) < tf(\mathbf{x}) \text{ for all } t > 1.$$

Add input, get CRtS: $F(z, \mathbf{x}) = zf(\mathbf{x}/z)$.

1.6.1 Elasticity of Scale

Let $y(t) = f(t\mathbf{x})$, then

$$\varepsilon(\mathbf{x}) \equiv \left. \frac{\frac{dy(t)}{y(t)}}{\frac{dt}{t}} \right|_{t=1} = \left. \frac{\partial \ln f(t\mathbf{x})}{\partial \ln t} \right|_{t=1}.$$

Technology exhibits locally (at $\mathbf{x}$) CR (DR,IR) tS if $\varepsilon(\mathbf{x}) = 1$ $(< 1, > 1)$.

Cobb-Douglas: $y = x_1^a x_2^b$,

$$f(tx_1, tx_2) = t^{a+b} f(x_1, x_2),$$
$$\varepsilon(\mathbf{x}) = \frac{df(tx_1, tx_2)}{dt} \frac{t}{f(tx_1, tx_2)} = a + b.$$

$f(\mathbf{x})$ is *homogeneous of degree* k if $f(t\mathbf{x}) = t^k f(\mathbf{x})$ for all $t > 0$.

Case $k = 0$: $f(t\mathbf{x}) = f(\mathbf{x})$;

Case $k = 1$: $f(t\mathbf{x}) = tf(\mathbf{x})$.

$h(\mathbf{x}) \equiv g(f(\mathbf{x}))$ is *homothetic* if g is strictly monotonic, and $f(\mathbf{x})$ is homogeneous of degree 1.

Isoquants look the same.