

4 Cost minimization

4.1 Problem of the firm.

$$c(\mathbf{w}, y) = \min_{\mathbf{x}} \mathbf{w}\mathbf{x}$$

s.t. $f(\mathbf{x}) = y$.

Lagrangian:

$$\mathcal{L}(\lambda, \mathbf{x}) = \mathbf{w}\mathbf{x} - \lambda(f(\mathbf{x}) - y).$$

FOC:

$$w_i - \lambda \frac{\partial f(\mathbf{x}^*)}{\partial x_i} = 0, \quad i = 1, \dots, n,$$
$$f(\mathbf{x}^*) = y.$$
$$\mathbf{w} = \lambda \mathbf{D}f(\mathbf{x}^*).$$

SOC: $f(x)$ is quasiconcave, $V(y)$ is convex suffice.
Somewhat complicated in general.

FOC imply:

$$\frac{w_i}{w_j} = \frac{\frac{\partial f(\mathbf{x}^*)}{\partial x_i}}{\frac{\partial f(\mathbf{x}^*)}{\partial x_j}}.$$

When $f(x_1, x_2)$ SOC becomes

$$\mathbf{h}^t \mathbf{D}^2 f(\mathbf{x}^*) \mathbf{h} \leq 0, \quad \forall \mathbf{h}, \mathbf{w}\mathbf{h} = 0.$$

4.1.1 Difficulties

- differentiability
- border solutions
- non-existence
- multiplicity

4.1.2 Examples

Cobb-Douglas:

$$c(\mathbf{w}, y) = \min_{x_1, x_2} w_1 x_1 + w_2 x_2, \\ \text{s.t. } x_1^a x_2^{1-a} = y.$$

FOC:

$$\frac{w_1}{w_2} = \frac{a}{1-a} \frac{x_2}{x_1}.$$

Express $x_1 = \frac{a}{1-a} \frac{w_2}{w_1} x_2$, substitute into f , obtain

$$x_2 = \left(\frac{1-a}{a} \right)^a \left(\frac{w_1}{w_2} \right)^a,$$

$$x_1 = \left(\frac{a}{1-a} \right)^{1-a} \left(\frac{w_2}{w_1} \right)^{1-a}.$$

$$c(w_1, w_2, y) = \left[\left(\frac{1-a}{a} \right)^a + \left(\frac{a}{1-a} \right)^{1-a} \right] w_1^a w_2^{1-a} y.$$

Leontief:

$$f(x_1, x_2) = \min [ax_1, bx_2], \\ c(w_1, w_2, y) = y \left(\frac{w_1}{a} + \frac{w_2}{b} \right).$$

Linear

$$c(w_1, w_2, y) = \min_{x_1, x_2} w_1 x_1 + w_2 x_2, \\ \text{s.t. } ax_1 + bx_2 = y, x_1 \geq 0, x_2 \geq 0.$$

Solution depends on $\frac{w_1}{w_2} \gtrless \frac{a}{b}$.

4.2 Weak Axiom of Cost Minimization

Observe $(\mathbf{w}^t, y^t, \mathbf{x}(\mathbf{w}^t, y^t))$ for some $t = 1, \dots, T$.

WACM: for all t and s ,

$$\mathbf{w}^t \mathbf{x}^t \leq \mathbf{w}^t \mathbf{x}^s.$$

Implications:

$$(\mathbf{w}^t - \mathbf{w}^s)(\mathbf{x}^t - \mathbf{x}^s) \leq 0 \quad \text{or} \quad \Delta \mathbf{w} \Delta x \leq 0.$$

RECOVERABILITY: VI, VO , Read the book.