

6 Utility maximization

6.1 Consumer Preferences

It is assumed that preferences are:

COMPLETE:

$$\forall x, y \in X, \quad x \succeq y \text{ or } y \succeq x \text{ or both.}$$

REFLEXIVE:

$$\forall x \in X, \quad x \succeq x.$$

TRANSITIVE:

$$\forall x, y, z \in X, \quad x \succeq y \text{ and } y \succeq z \implies x \succeq z.$$

\succ and \sim are defined obviously.

CONTINUITY:

$\forall y \in X$, sets $\{x : x \succeq y\}$ and $\{x : x \preceq y\}$ are closed.

WEAK MONOTONICITY:

$$x \geq y \implies x \succeq y.$$

STRONG MONOTONICITY:

$$x \geq y \text{ and } x \neq y \implies x \succ y.$$

LOCAL NONSATIATION:

$\forall x \in X, \forall \varepsilon > 0, \exists y \in X, |x - y| < \varepsilon$ that $y \succ x$.

CONVEXITY and STRICT CONVEXITY: as usual.

6.2 Existence of a utility function.

Theorem: Suppose preferences are COMP, REFL, TRAN, CONT, and SMON. Then continuous

$$u : \mathbb{R}_+^k \rightarrow \mathbb{R},$$

that represents these preferences, *exists*.

Proof: $\forall \mathbf{x} \in X$ define $u(\mathbf{x})$ as

$$\mathbf{x} \sim u(\mathbf{x})\mathbf{e}.$$

Define: *MRS (Marginal Rate of Substitution): ...*

NOTE: MRS is invariant to monotonic transformations of utility functions:

$$g(\mathbf{x}) = v(u(\mathbf{x})), \text{ where } v' > 0.$$

6.3 Consumer behavior

Suppose $\mathbf{p} = (p_1, \dots, p_n)$ is vector of prices for goods, and m is money holdings of a consumer.

Define

$$B = \{\mathbf{x} \in X : \mathbf{p}\mathbf{x} \leq m\}.$$

Consumer problem:

$$\begin{aligned} & \max u(\mathbf{x}) \\ & \text{s.t. } \mathbf{x} \in X, \mathbf{p}\mathbf{x} \leq m. \end{aligned}$$

Solution:

$$\begin{aligned} & \mathbf{x}(\mathbf{p}, m) \text{—} \textit{Marshallian demand}, \\ & v(\mathbf{p}, m) = u(\mathbf{x}(\mathbf{p}, m)) \text{—} \textit{indirect utility function}. \end{aligned}$$

Brief analysis: Usual problems; also $p_i = 0$ with possibility of $x_i^* = \infty$.

6.4 Indirect utility

Properties: $v(\mathbf{p}, m)$ is

1. Nonincreasing in \mathbf{p} ; nondecreasing in m .
2. Homogeneous of degree 0 in (\mathbf{p}, m) .
3. Quasiconvex in \mathbf{p} ; or $\{\mathbf{p} : v(\mathbf{p}, m) \leq k\}$ is a convex set for all k .
4. Continuous at all $\mathbf{p} \gg \mathbf{0}, m > 0$.

6.5 Expenditure function

Expenditure minimization (*a la* cost minimization)

$$\begin{aligned} e(\mathbf{p}, u) &= \min \mathbf{p}\mathbf{x} \\ \text{s.t. } &u(\mathbf{x}) \geq u. \end{aligned}$$

Properties: Exactly as for the cost function, which are ...

Hicksian demand:

$$\mathbf{h}(\mathbf{p}, u) = \left(\frac{\partial e(\mathbf{p}, u)}{\partial p_i} \right)_{i=1}^n.$$

Properties: ...

6.6 Important Identities

$$e(\mathbf{p}, v(\mathbf{p}, m)) \equiv m,$$

$$v(\mathbf{p}, e(\mathbf{p}, u)) \equiv u;$$

$$x_i(\mathbf{p}, m) \equiv h_i(\mathbf{p}, v(\mathbf{p}, m)),$$

$$h_i(\mathbf{p}, u) \equiv x_i(\mathbf{p}, e(\mathbf{p}, u)).$$

Roy's identity:

$$x_i(\mathbf{p}, m) = -\frac{\frac{\partial v(\mathbf{p}, m)}{\partial p_i}}{\frac{\partial v(\mathbf{p}, m)}{\partial m}} \text{ for all } i = 1, \dots, k.$$

Money metric utilities:

$$\text{direct} : m(\mathbf{p}, \mathbf{x}) = e(\mathbf{p}, u(\mathbf{x})),$$

$$\text{indirect} : \mu(\mathbf{p}; \mathbf{q}, m) = e(\mathbf{p}, v(\mathbf{q}, m)).$$