1 Auctions

- Ancient "market" mechanisms. Widespread in use. A lot of varieties.
- Simple and transparent games (mechanisms). Universal rules (does not depend on the object for sale), anonymous (all bidders are treated equally).
- Operate well in the incomplete information environments. Seller (and sometimes bidders as well) does not know how the others value the object.
- Optimality and efficiency in broad range of settings.
- Probably the most active area of research in economics.

1.1 Notation (Symmetric IPV)

Independent private values setting with symmetric risk-neutral buyers, no budget constraints.

- Single indivisible object for sale.
- N potential buyers, indexed by i. N commonly known to all bidders.
- X_i — valuation of buyer i — maximum willingness to pay for the object.
- $X_i \sim F[0, \omega]$ with continuous $f = F'$ and full support.
- X_i is private value (signal); all X_i are iid, which is common knowledge.

1.2 Common auctions

SEALED-BID Auctions.

- First price sealed-bid auction:

  Each bidder submits a bid $b_i \in \mathbb{R}$ (sealed, or unobserved by the others). The winner is the buyer with the highest bid, the winner pays her bid.

- Second price sealed-bid auction:

  As above, the winner pays second highest bid — highest of the bids of the others.

- Kth price auction:

  The winner pays the Kth highest price.

- All-pay auction:

  All bidders pay their bids.

OPEN (DYNAMIC) Auctions.

- Dutch auction:

  The price of the object starts at some high level, when no bidder is willing to pay for it. It is decreased until some bidder announces his willingness to buy. He obtains the object at this price.

  Note: Dutch and First-price auctions are equivalent in strong sense.

- English auction:

  The price of the object starts at zero and increases. Bidders start active — willing to buy the object at a price of zero. At a given price, each bidder is either willing to buy the object at that price (active) or not (inactive). While the price is increasing, bidders reduce(*) their demands. The auction stops when only one bidder remains active. She is the winner, pays the price at which the last of the others stopped bidding.

  Note: English auction is in a weak sense equivalent to the second-price auction.

1.3 First-price auction

Payoffs

$$\Pi_i = \begin{cases} x_i - b_i, & \text{if } b_i > \max_{j \neq i} b_j, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition: Symmetric equilibrium strategies in a first-price auction are given by

$$\beta^{\mathsf{I}}(x) = E\left[Y_1 | Y_1 < x\right],$$

where $Y_1 = \max_{j \neq i}\{X_j\}$.

Proof: Easy to check that it is eq.strat., let us derive it.

Suppose every other bidder except i follows strictly increasing (and differentiable) strategy $\beta(x)$.

Equilibrium trade-off: Gain from winning versus probability of winning.

Expected payoff from bidding b when receiving x_i is

$$G_{Y_1}(\beta^{-1}(b)) \times (x_i - b).$$

FOC:

$$\frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))}(x-b) - G(\beta^{-1}(b)) = 0.$$

In symmetric equilibrium, $b(x) = \beta(x)$, so FOC $\Rightarrow$

$$\begin{aligned} G(x)\beta'(x) + g(x)\beta(x) &= xg(x), \\ \frac{d}{dx}(G(x)\beta(x)) &= xg(x), \\ \beta(x) &= \frac{1}{G(x)}\int_0^x yg(y)dy, \\ &= E[Y_1 | Y_1 < x]. \end{aligned}$$

In the first price auction expected payment is

$$\begin{aligned} m^{\mathrm{I}}(x) &= \Pr[\text{Win}] \times b(x) \\ &= G(x) \times E[Y_1 | Y_1 < x]. \end{aligned}$$

1.4 Examples:

1. Suppose values are uniformly distributed on $[0,1]$.

   $F(x) = x$, then $G(x) = x^{N-1}$ and

   $$\beta^{\mathrm{I}}(x) = \frac{N-1}{N}x.$$

2. Suppose values are exponentially distributed on $[0,\infty)$.

   $F(x) = 1 - e^{-\lambda x}$, for some $\lambda > 0$ and $N = 2$, then

   $$\begin{aligned} \beta^{\mathrm{I}}(x) &= x - \int_0^x \frac{F(y)}{F(x)}dy \\ &= \frac{1}{\lambda} - \frac{xe^{-\lambda x}}{1-e^{-\lambda x}}. \end{aligned}$$

   Note that if, say for $\lambda = 2$, x is very large the bid would not exceed 50 cents.

1.5 Second-price auction

Proposition: In a second-price sealed-bid auction, it is a weakly dominant strategy to bid

$$\beta^{\mathsf{II}}(x) = x.$$

In the second price auction expected payment of the winner with value x is the expected value of the second highest bid given x, which is the expectation of the second-highest value given x.

Thus, expected payment in the second-price auction is

$$\begin{aligned} m^{\mathsf{I}}(x) &= \mathsf{Pr[Win]} \times E\left[Y_1 | Y_1 < x\right] \\ &= G(x) \times E\left[Y_1 | Y_1 < x\right]. \end{aligned}$$