MIT 14.04 Intermediate Microeconomic Theory

Fall 2003

Answer keys for problem set 2

Exercise 4.4

If the first plant wants to produce y, its cost will be:

$$C_1(y) = K(w_1, w_2, a)y = \min_{x_1^a x_2^{1-a} \geq y} w_1 x_1 + w_2 x_2$$

where $K(w_1, w_2, a) = a^{-a}(1-a)^{a-1} w_1^a w_2^{1-a}$.

The cost of the second plant is symmetric. To allocate the production between its two plants, the firm solves:

$$C(y) = \min_{y'+y'' \geq y} C_1(y') + C_2(y'') = \min_{y'+y'' \geq y} K(w_1, w_2, a)y' + K(w_1, w_2, b)y''$$

Note that both the constraint and the cost function are linear in y' and y'', hence the solution is a corner solution not an interior one.

$$\begin{aligned}
if\ K(w_1, w_2, a) < K(w_1, w_2, b) &\Rightarrow\ y' = y\ and\ y'' = 0,\ C(y) = C_1(y) \\
if\ K(w_1, w_2, a) > K(w_1, w_2, b) &\Rightarrow\ y' = 0\ and\ y'' = y,\ C(y) = C_2(y) \\
if\ K(w_1, w_2, a) = K(w_1, w_2, b) &\Rightarrow\ y' \in [0, y]\ and\ y'' = y - y',\ C(y) = C_1(y) = C_2(y)
\end{aligned}$$

Exercise 4.6

The firm solves:

$$\begin{aligned}
\min \quad & c_1(y_1) + c_2(y_2) \\
y_1 + y_2 &\geq y \\
y_1 &\geq 0 \\
y_2 &\geq 0
\end{aligned}$$

Set up the Lagrangian:

$$L = c_1(y_1) + c_2(y_2) - \lambda(y_1 + y_2 - y) - \mu_1 y_1 - \mu_2 y_2$$

where λ, μ_1 and μ_2 are non-negative. The Kuhn-Tucker conditions are:

$$\frac{2}{\sqrt{y_1}} - \lambda - \mu_1 = 0$$

$$\begin{aligned}\frac{1}{\sqrt{y_2}} - \lambda - \mu_2 &= 0\\ \lambda(y_1 + y_2 - y) &= 0\\ \mu_1 y_1 &= 0\\ \mu_2 y_2 &= 0\end{aligned}$$

- Interior solution: both μ_i are equal to 0. This implies that

$$\lambda = \frac{2}{\sqrt{y_1}} = \frac{1}{\sqrt{y_2}} > 0$$

$\lambda > 0$ implies that $y_1 + y_2 = y$. Solving for y_i, you find:

$$y_1 = \frac{y}{5},\ y_2 = \frac{4y}{5}\ \textit{and}\ c(y) = 2\sqrt{5}\sqrt{y}$$

- Corner solution: $\mu_1 > 0$ and $\mu_2 = 0$.

$$y_1 = 0,\ y_2 = y\ \textit{and}\ c(y) = 2\sqrt{y}$$

As the corner solution is better than the interior one (which indeed is a maximum and not a minimum), the firm will produce using solely its plant #2. Note that it is no use of looking at the other corner solution as $c_1(y) = 2c_2(y)$, if the firm decides to use a single plant it will use only the second one. Note that it is not enough to have $c_1(y) = 2c_2(y)$ to conclude that the firm produces using on plant only. (you can check that with $c_1(y) = 2y^2$ and $c_1(y) = y^2$).

Exercise 4.8

The cost function is:

$$c(w_1, w_2, y) = 2w_1^{\frac{1}{2}} w_2^{\frac{1}{2}} y^{\frac{1}{2}}$$

$$c(1, 1, y) = 4 \Rightarrow y = 4$$

Exercise 5.4

Start by drawing the production function, then you can read the solution on the graph. See figure 1

$if\ \frac{w_1}{w_2} > 2,$	$x_1 = 0$	$x_2 = y$	$c = w_2 y$
$if\ \frac{w_1}{w_2} = 2,$	$x_1 \in [0, \frac{y}{3}]$	$x_2 = y - 2x_1$	$c = w_2 y = \frac{w_1}{2} y$
$if\ 2 > \frac{w_1}{w_2} > \frac{1}{2},$	$x_1 = \frac{y}{3}$	$x_2 = \frac{y}{3}$	$c = (w_1 + w_2)\frac{y}{3}$
$if\ \frac{w_1}{w_2} = \frac{1}{2},$	$x_1 \in [\frac{y}{3}, y]$	$x_2 = \frac{y - x_1}{2}$	$c = w_1 y = \frac{w_2}{2} y$
$if\ \frac{1}{2} > \frac{w_1}{w_2},$	$x_1 = y$	$x_2 = 0$	$c = w_1 y$

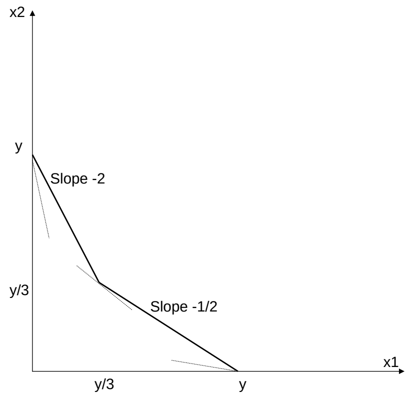

Figure 1: Production function composed of two linear parts. Three examples of budget lines are drawn to illustrate the optimal points.

Exercise 5.6

- x_1 has to be homogeneous of degree 0 $\Rightarrow a = \frac{1}{2}$.
- x_2 has to be homogeneous of degree 0 $\Rightarrow c = -\frac{1}{2}$.
- The substitution matrix has to be symmetric $\frac{\partial x_1}{\partial w_2} = \frac{\partial x_2}{\partial w_1} \Rightarrow b = 3$.

Exercise 5.8

If the firm produces, what is the best it could do?

$$\frac{p^2}{4} - 1 = \max_y py - (y^2 + 1)$$

indeed FOC: $y = \frac{p}{2}$ and SOC: $-2 < 0$.
But the firm has also the option of not producing which leads to zero profit.

- if $p > 2$, $y = \frac{p}{2}$ and $\pi = \frac{p^2}{4} - 1$
- if $p = 2$, $y = \frac{p}{2}$ or $y = 0$ and $\pi = 0$
- if $p < 2$, $y = 0$ and $\pi = 0$

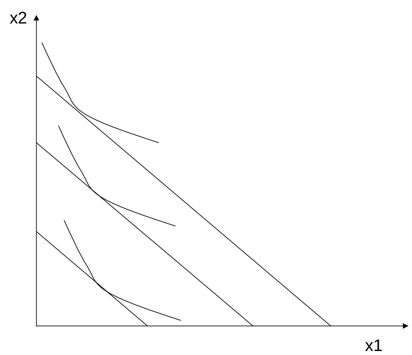

Figure 2: x_1 is an inferior good.

Exercise 5.12

(a) see figure 2

(b) Let's start by showing a general result for an interior solution when the production function is CRtS. (easily checked on a graph)

$$x(w, ty) = tx(w, y)$$

The demand functions are characterized by:

$$\begin{aligned} f(x(w,y)) &= y \\ \frac{\frac{\partial f}{\partial x_i}(x(w,y))}{\frac{\partial f}{\partial x_j}(x(w,y))} &= \frac{w_i}{w_j} \end{aligned}$$

f is CRtS means it is homogeneous of degree 1, which implies that its partial derivatives are homogeneous of degree 0.

$$\begin{aligned} f(tx(w,y)) &= tf(x(w,y)) = ty \\ \frac{\frac{\partial f}{\partial x_i}(tx(w,y))}{\frac{\partial f}{\partial x_j}(tx(w,y))} &= \frac{\frac{\partial f}{\partial x_i}(x(w,y))}{\frac{\partial f}{\partial x_j}(x(w,y))} = \frac{w_i}{w_j} \end{aligned}$$

therefore $x(w, ty) = tx(w, y)$. Differentiate w.r.t. t and set $t = 1$ to get $\frac{\partial x_i}{\partial y}(w, y) = \frac{x_i(w,y)}{y} \geq 0$. No factor can be inferior.

(c) Assume $\frac{\partial^2 c}{\partial y \partial w_i} < 0$ i.e. the marginal cost decreases when the price w_i increases. Use Shephard's lemma $x_i = \frac{\partial c}{\partial w_i}$ to conclude that $\frac{\partial x_i}{\partial y} < 0$, x_i is inferior.

Exercise 5.16

1. c is not homogeneous of degree 1 as $c(tw, y) = t^{1.5} c(w, y)$.

2. - c is homogeneous of degree 1 as $c(tw, y) = tc(w, y)$.

   - c is non decreasing as $\frac{\partial c}{\partial w_i} = y(1 + 0.5\sqrt{\frac{w_j}{w_i}}) > 0$.

   - c is concave as $c(tw + (1-t)w', y) \geq tc(w, y) + (1-t)c(w', y) \Leftrightarrow \sqrt{(tw_1 + (1-t)w_2)(tw'_1 + (1-t)w'_2)} \geq t\sqrt{w_1 w_2} + (1-t)\sqrt{w'_1 w'_2} \Leftrightarrow t(1-t)(\sqrt{w_1 w'_2} + \sqrt{w'_1 w_2})^2 \geq 0$.

   Note: as it was suggested in class, checking that the Hessian matrix is negative semi-definite is also valid.

   - c is continuous.

   The idea for going from the cost function back to the production function is to use the demand functions to eliminate the prices and find a relation linking the output and the inputs. The demand functions are $x_i = y(1 + .5\sqrt{\frac{w_j}{w_i}})$. Extract $\frac{w_1}{w_2}$ from the demand functions in order to eliminate the ws and get a relation between y, x_1 and x_2 only. Then rearrange this relation to get the production function.

$$\begin{aligned} \sqrt{\frac{w_1}{w_2}} &= 2(\frac{x_2}{y} - 1) = \left(2(\frac{x_1}{y} - 1)\right)^{-1} \\ \Rightarrow y &= \frac{2}{3}(x_1 + x_2) - \sqrt{\frac{4}{9}(x_1 + x_2)^2 - \frac{4}{3}x_1 x_2} \end{aligned}$$

   You initially get two candidates for the production function which are the two roots. Knowing that it must be concave in x_i allows to select one of them.

3. c is not homogeneous

4. c is not monotonic as $\frac{\partial c}{\partial w_i} = y\left(1 - \frac{1}{2}\sqrt{\frac{w_j}{w_i}}\right) < 0$ when $4w_i < w_j$.

5. c is not increasing in y.