# 2 Profit maximization

FIRMS MAXIMIZE PROFITS

$$\text{Revenue} - \text{Cost} \to \max_{\text{activities}} .$$

Marginal analysis: $\frac{\partial R}{\partial a_i} = \frac{\partial C}{\partial a_i}$.

Internal constraints: Technology.

External constraints: Market conditions.

Price-taking behavior:

Small, competitive firm.

## 2.1 Problem of the firm

Profit function:

$$\begin{aligned} \pi(\mathbf{p}) &= \max_{\mathbf{y} \in Y} \mathbf{p}\mathbf{y}, \\ \pi(p, \mathbf{w}) &= pf(\mathbf{x}) - \mathbf{w}\mathbf{x} \to \max_{\mathbf{x}} . \end{aligned}$$

Cost function:

$$c(\mathbf{w}, y) = \min_{\mathbf{x} \in V(y)} \mathbf{w}\mathbf{x}.$$

FOC (profits):

$$p\frac{\partial f(\mathbf{x}^*)}{\partial x_i} = w_i, \quad i = 1, \ldots, n,$$

$$p\mathbf{D}f(\mathbf{x}^*) = \mathbf{w};$$

$$\mathbf{D}f(\mathbf{x}^*) = \left( \frac{\partial f(\mathbf{x}^*)}{\partial x_1}, \ldots, \frac{\partial f(\mathbf{x}^*)}{\partial x_n} \right).$$

SOC:

$$\forall \mathbf{h} \in \mathbb{R}^n \quad \mathbf{h}\mathbf{D}^2 f(\mathbf{x}^*)\mathbf{h}^t \leq 0,$$
$$\mathbf{D}^2 f(\mathbf{x}^*) = \left(\frac{\partial^2 f(\mathbf{x}^*)}{\partial x_i \partial x_j}\right).$$

Factor demand function: $\mathbf{x}(p, \mathbf{w}) \equiv \mathbf{x}^*(p, \mathbf{w})$.

Supply function: $y(p, \mathbf{w}) = f(\mathbf{x}(p, \mathbf{w}))$.

### 2.1.1 Difficulties

- Non-differentiability (Leontief);
- Boundary solutions $\Longrightarrow$ Kuhn-Tucker;
- Existence (CRtS, IRtS).

### 2.1.2 Cobb-Douglas

Technology:

$$f(x_1, x_2) = x_1^a x_2^b.$$

Firm's problem:

$$\pi(p, w_1, w_2) = \max_{x_1, x_2} pf(\mathbf{x}) - w_1 x_1 - w_2 x_2.$$

FOC:

$$p\frac{\partial f(\mathbf{x}^*)}{\partial x_1} = pa\frac{f(\mathbf{x}^*)}{x_1} = w_1, \quad p\frac{\partial f}{\partial x_2} = pb\frac{f(\mathbf{x}^*)}{x_2} = w_2.$$

SOC:

$$\mathbf{D}^2 f(\mathbf{x}^*) = x_1^{a-2} x_2^{b-2} \begin{pmatrix} a(a-1)x_2^2 & abx_1x_2 \\ abx_1x_2 & b\,(b-1)\,x_1^2 \end{pmatrix},$$

$a < 1, \quad b < 1, \quad a + b < 1 \Longrightarrow \mathbf{D}^2 f(\mathbf{x}^*)$ is neg. def.

Solution:

$$x_1^* = \left[p\left(\frac{a}{w_1}\right)^{1-b}\left(\frac{b}{w_2}\right)^{b}\right]^{\frac{1}{1-a-b}},$$

$$x_2^* = \left[p\left(\frac{a}{w_1}\right)^{a}\left(\frac{b}{w_2}\right)^{1-a}\right]^{\frac{1}{1-a-b}},$$

$$y^* = \left[p^{a+b}\frac{a^a b^b}{w_1^a w_2^b}\right]^{\frac{1}{1-a-b}}.$$

All $x_1^*$, $x_2^*$, and $y^*$ are homogeneous of degree 0.

Comparative statics: response to changes.

Envelope theorem:

$$\frac{\partial \pi^*}{\partial p} = y^*, \quad \frac{\partial \pi^*}{\partial w_i} = x_i^*.$$

Factor demand slope:

$$w_i \uparrow \implies x_i^*, x_j^* \downarrow .$$

## 2.2 Properties of demand and supply

- $\mathbf{x}(p, \mathbf{w})$, $y(p, \mathbf{w})$ are homogeneous of degree 0.

Single input: $pf(x) - wx \to \max_x$.

$$\begin{aligned}
(\text{FOC}) &: \quad pf'(x(p,w)) - w = 0,\\
\frac{\partial \text{FOC}}{\partial w} &: \quad pf''(x(p,w))\frac{\partial x(p,w)}{\partial w} - 1 = 0,\\
(\text{SOC}) &: \quad pf''(x(p,w)) \le 0,\\
&\implies \quad \frac{\partial x(p,w)}{\partial w} \le 0.
\end{aligned}$$

$x \in \mathbb{R}^n$: normalize $p = 1$.

$$\begin{aligned}
(\text{FOC}) &: \quad \mathbf{D}f(\mathbf{x}) - \mathbf{w} = \mathbf{0},\\
\frac{\partial \text{FOC}}{\partial \mathbf{w}} &: \quad \mathbf{D}^2 f(\mathbf{x})\mathbf{D}(\mathbf{x}(\mathbf{w})) - \mathbf{I}_n = \mathbf{0}.
\end{aligned}$$

$\mathbf{D}(\mathbf{x}(\mathbf{w})) \equiv \left(\frac{\partial x_i(\mathbf{w})}{\partial w_j}\right)$—substitution matrix.

- $\mathbf{D}(\mathbf{x}(\mathbf{w})) = \left[\mathbf{D}^2 f(\mathbf{x})\right]^{-1}$—symmetric neg. def.

$\mathbf{w} \to \mathbf{w} + d\mathbf{w}$, then $d\mathbf{x} = \mathbf{D}(\mathbf{x}(\mathbf{w}))d\mathbf{w}^t$. So,

- $d\mathbf{w}\ d\mathbf{x} = d\mathbf{w}\mathbf{D}(\mathbf{x}(\mathbf{w}))d\mathbf{w}^t \leq 0$.

## 2.3 Weak Axiom of Profit Maximization

Observe $\left(\mathbf{p}^t, \mathbf{y}(\mathbf{p}^t)\right)$ for some $t = 1, \ldots, T$.

WAPM: for all $t$ and $s$,

$$\mathbf{p}^t\mathbf{y}^t \geq \mathbf{p}^t\mathbf{y}^s.$$

Implications:

$$\left(\mathbf{p}^t - \mathbf{p}^s\right)\left(\mathbf{y}^t - \mathbf{y}^s\right) \geq 0 \quad \text{or} \quad \Delta\mathbf{p}\Delta\mathbf{y} \geq \mathbf{0}.$$

RECOVERABILITY: Read the book.